Exercise 13

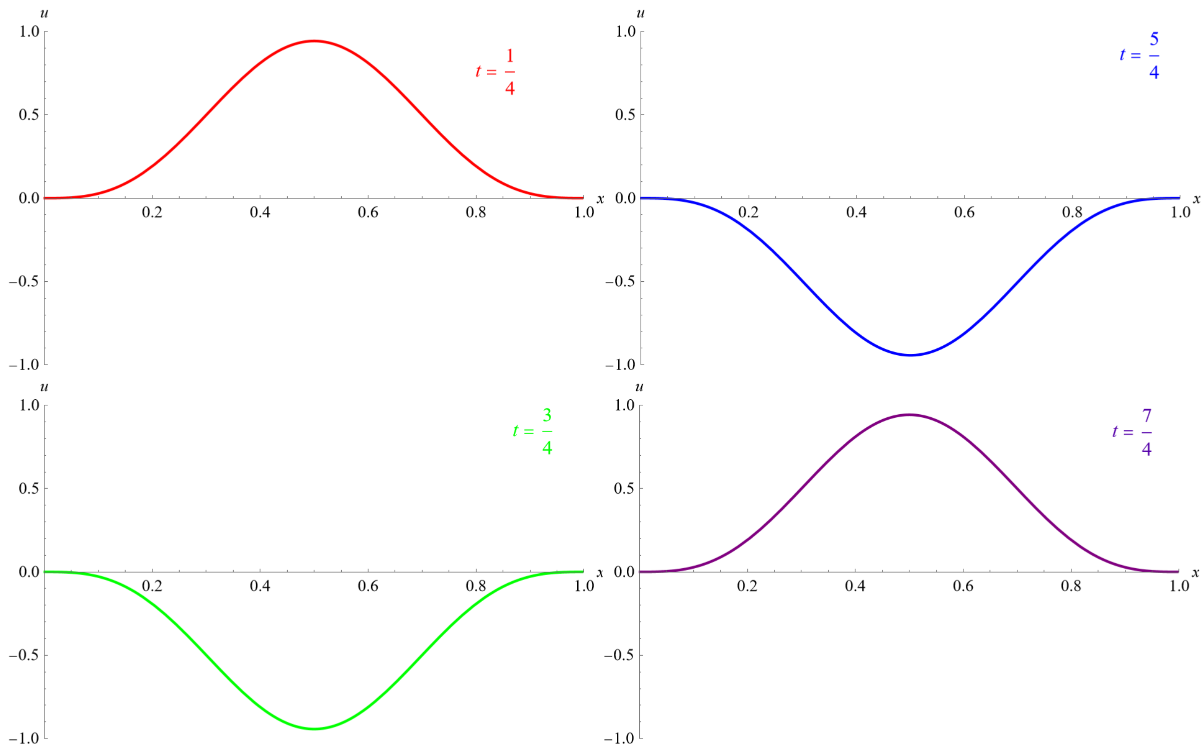
Referring to Figure 8, explain why frames 2, 4, 6, and 8 appear to be symmetric about $x = 1/2$.

Solution

Figure 8 shows the graph of

$$u(x,t) = \sin \pi x \cos \pi t - \frac{1}{2} \sin 2\pi x \cos 2\pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t, \quad 0 < x < 1,$$

a solution to the wave equation on the interval $0 < x < L$ with fixed ends, versus x at several times. The graphs in frames 2, 4, 6, and 8 are shown below.



The graphs do seem to be symmetric about $x = 1/2$, and they occur at $t = 1/4$, $t = 3/4$, $t = 5/4$, and $t = 7/4$. Try plugging in $t = (2n+1)/4$, where $n = 0, 1, 2, \ldots$, into the formula for u.

$$\begin{aligned} u\left(x, \frac{2n+1}{4}\right) &= \sin \pi x \cos\left[\pi\left(\frac{2n+1}{4}\right)\right] - \frac{1}{2} \sin 2\pi x \cos\left[2\pi\left(\frac{2n+1}{4}\right)\right] + \frac{1}{3} \sin 3\pi x \cos\left[3\pi\left(\frac{2n+1}{4}\right)\right] \\ &= \sin \pi x \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) - \frac{1}{2} \sin 2\pi x \cos\left(n\pi + \frac{\pi}{2}\right) + \frac{1}{3} \sin 3\pi x \cos\left(\frac{3n\pi}{2} + \frac{3\pi}{4}\right) \\ &= \sin \pi x \left(\cos \frac{n\pi}{2} \cos \frac{\pi}{4} - \sin \frac{n\pi}{2} \sin \frac{\pi}{4}\right) - \frac{1}{2} \sin 2\pi x \left(\cos n\pi \underbrace{\cos \frac{\pi}{2}}_{=0} - \underbrace{\sin n\pi}_{=0} \sin \frac{\pi}{2}\right) \\ &\quad + \frac{1}{3} \sin 3\pi x \left(\cos \frac{3n\pi}{2} \cos \frac{3\pi}{4} - \sin \frac{3n\pi}{2} \sin \frac{3\pi}{4}\right) \end{aligned}$$

Continue the simplification.

$$\begin{aligned} u\left(x, \frac{2n+1}{4}\right) &= \sin \pi x \left(\frac{1}{\sqrt{2}} \cos \frac{n\pi}{2} - \frac{1}{\sqrt{2}} \sin \frac{n\pi}{2}\right) + \frac{1}{3} \sin 3\pi x \left(-\frac{1}{\sqrt{2}} \cos \frac{3n\pi}{2} - \frac{1}{\sqrt{2}} \sin \frac{3n\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}} \sin \pi x \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) - \frac{1}{3\sqrt{2}} \sin 3\pi x \left(\cos \frac{3n\pi}{2} + \sin \frac{3n\pi}{2}\right) \end{aligned}$$

Now replace x with $x + \frac{1}{2}$ to shift the graph to the left by 1/2 units.

$$\begin{aligned} u\left(x + \frac{1}{2}, \frac{2n+1}{4}\right) &= \frac{1}{\sqrt{2}} \sin \left[\pi \left(x + \frac{1}{2}\right)\right] \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) - \frac{1}{3\sqrt{2}} \sin \left[3\pi \left(x + \frac{1}{2}\right)\right] \left(\cos \frac{3n\pi}{2} + \sin \frac{3n\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}} \sin \left(\pi x + \frac{\pi}{2}\right) \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) - \frac{1}{3\sqrt{2}} \sin \left(3\pi x + \frac{3\pi}{2}\right) \left(\cos \frac{3n\pi}{2} + \sin \frac{3n\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}} \left(\sin \pi x \underbrace{\cos \frac{\pi}{2}}_{=0} + \cos \pi x \underbrace{\sin \frac{\pi}{2}}_{=1}\right) \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) \\ &\qquad - \frac{1}{3\sqrt{2}} \left(\sin 3\pi x \underbrace{\cos \frac{3\pi}{2}}_{=0} + \cos 3\pi x \underbrace{\sin \frac{3\pi}{2}}_{=-1}\right) \\ &= \frac{1}{\sqrt{2}} (\cos \pi x) \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) - \frac{1}{3\sqrt{2}} (-\cos 3\pi x) \\ &= \frac{1}{\sqrt{2}} \cos \pi x \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right) + \frac{1}{3\sqrt{2}} \cos 3\pi x \end{aligned}$$

This is an even function because $\cos \pi x$ and $\cos 3\pi x$ are even functions. Therefore, $u\left(x, \frac{2n+1}{4}\right)$ is symmetric about $x = 1/2$.